MATHEMATICS

ASSOCIATIVE ALGEBRAS WITH INVOLUTION AND JORDAN ALGEBRAS

BY

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Dedicated to Professor Hans Freudenthal on his sixtieth birthday

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If $\mathfrak{A}$ is an associative algebra with involution J over a field of characteristic $\neq 2$ then $\mathfrak{A}$ determines the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ of J-symmetric elements relative to the Jordan product $a \cdot b = \frac{1}{2}(ab+ba)$. We obtain in this way a functor from the category of associative algebras with involution into the category of Jordan algebras. On the other hand, if $\mathfrak{J}$ is a Jordan algebra, then $\mathfrak{J}$ determines an associative algebra with involution $(\mathrm{su}\,(\mathfrak{J}), \pi)$ where $\mathrm{su}\,(\mathfrak{J})$ is the special universal envelope (= universal associative algebra for special representations) and π is the main involution in $\mathrm{su}\,(\mathfrak{J})$. This defines a functor S from the category of Jordan algebras into the category of associative algebras with involution.

If $\mathfrak{J}$ is a Jordan algebra then we have a natural homomorphism s of $\mathfrak{J}$ into $\mathfrak{H}(\mathrm{su}\,(\mathfrak{J}), \pi)$ and if $(\mathfrak{A}, J)$ is an associative algebra with involution then we have a natural homomorphism h of $(\mathrm{su}\,(\mathfrak{H}(\mathfrak{A}, J), \pi)$ into $(\mathfrak{A}, J)$. It is immediate that s is injective if and only if $\mathfrak{J}$ is special and we call $\mathfrak{J}$ reflexive if s is surjective. The homomorphism h is surjective if and only if the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{H}(\mathfrak{A}, J)$ is $\mathfrak{A}$. We call the associative algebra with involution $(\mathfrak{A}, J)$ perfect if h is an isomorphism of $(\mathrm{su}\,\mathfrak{H}(\mathfrak{A}, J), \pi)$ (onto $(\mathfrak{A}, J)$). In § 2 we shall obtain some sufficient conditions for reflexivity of Jordan algebras and for perfection of associative algebras with involution. In § 3 we shall apply the general notions and one of the general results of § 2 to obtain an improved formulation and derivation of the main theorem on the classification of special finite dimensional simple Jordan algebras (cf. F. D. JACOBSON and N. JACOBSON [1]).

1. *Basic notions*. Throughout this paper "algebra" will mean not necessarily associative algebra with an identity element 1 over a field of characteristic $\neq 2$. The usual conventions for algebras with 1 will be adopted: subalgebras contain 1, homomorphisms map 1 into 1, etc. By an *associative algebra with involution* we mean a pair $(\mathfrak{A}, J)$ where $\mathfrak{A}$ is an associative algebra and J is an involution in A, that is, J is an anti-

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automorphism in $\mathfrak{A}$ such that $J^2=1$, the identity mapping. An *ideal* $\mathfrak{K}$ of $(\mathfrak{A}, J)$ is an ideal of $\mathfrak{A}$ such that $\mathfrak{K}^J \subseteq \mathfrak{K}$ and a *homomorphism* η of the associative algebra with involution $(\mathfrak{A}, J)$ into the associative algebra with involution $(\mathfrak{B}, K)$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $J\eta=\eta K$. The class of associative algebras with involution over a fixed field Φ together with the homomorphisms as morphisms is a category which we shall denote as cat AI/Φ.

Let $(\mathfrak{A}, J)$ be an associative algebra with involution over the field Φ and let $\mathfrak{H}(\mathfrak{A}, J)$ denote the set of J-symmetric elements $(a^J=a)$ of $\mathfrak{A}$. Then $\mathfrak{H}(\mathfrak{A}, J)$ is a subspace of $\mathfrak{A}$ closed under the composition $a\cdot b=\frac{1}{2}(ab+ba)$. We recall that if $\mathfrak{A}$ is any associative algebra then $\mathfrak{A}$ determines a Jordan algebra $\mathfrak{A}^+$ whose underlying vector space is the same as that of $\mathfrak{A}$ and whose multiplication composition is the Jordan product $a\cdot b=\frac{1}{2}(ab+ba)$. We recall also that an algebra $\mathfrak{J}$ over a field Φ is a Jordan algebra if its multiplication composition $a\cdot b$ satisfies the identities: $a\cdot b=b\cdot a$, $(a^{\cdot 2}\cdot b)\cdot a=a^{\cdot 2}\cdot(b\cdot a)$ where $a^{\cdot 2}=a\cdot a$. Let cat J/Φ denote the category of Jordan algebras over the field Φ with the morphisms as homomorphisms. Then the result we noted before is that any associative algebra with involution $(\mathfrak{A}, J)$ determines a Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$. Next let η be a homomorphism of $(\mathfrak{A}, J)$ into the associative algebra with involution $(\mathfrak{B}, K)$. Then it is clear that η maps $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{H}(\mathfrak{B}, K)$ and that the restriction $\eta|\mathfrak{H}$ of η to $\mathfrak{H}(\mathfrak{A}, J)$ is a homomorphism of the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{H}(\mathfrak{B}, K)$. It is immediate that the mappings $(\mathfrak{A}, J)\to\mathfrak{H}(\mathfrak{A}, J)$, $\eta\to\eta|\mathfrak{H}$ where η is a homomorphism of $(\mathfrak{A}, J)$ into $(\mathfrak{B}, K)$ define a functor H of cat AI/Φ into cat J/Φ.

If $\mathfrak{A}$ is an algebra over Φ and P is an extension field of Φ then $\mathfrak{A}_P=P\otimes_\Phi\mathfrak{A}$ is an algebra over P. If $(\mathfrak{A}, J)$ is an associative algebra with involution then the linear mapping J in $\mathfrak{A}$ has a unique extension to a linear mapping J in $\mathfrak{A}_P$. The pair $(\mathfrak{A}_P, J)$ is an associative algebra with involution over P. We have an isomorphism of the Jordan algebra $\mathfrak{H}(\mathfrak{A}_P, J)$ with $\mathfrak{H}(\mathfrak{A}, J)_P$ and we can identify these two Jordan algebras.

We proceed next to define a functor of the category cat J/Φ of Jordan algebras over Φ into cat AI/Φ. If $\mathfrak{J}$ is a Jordan algebra over Φ and $\mathfrak{A}$ is an associative algebra over Φ then we define an *associative specialization* σ of $\mathfrak{J}$ in $\mathfrak{A}$ to be a homomorphism of $\mathfrak{J}$ into the Jordan algebra $\mathfrak{A}^+$. A *special universal envelope* of $\mathfrak{J}$ is a pair $(\mathfrak{U}, s)$, where $\mathfrak{U}$ is an associative algebra and s is an associative specialization of $\mathfrak{J}$ in $\mathfrak{U}$, such that if σ is any associative specialization of $\mathfrak{J}$ in $\mathfrak{A}$, then there exists a unique homomorphism η of $\mathfrak{U}$ into $\mathfrak{A}$ such that the following diagram is commutative:

(1)
$$\begin{tikzcd} \mathfrak{J} \arrow[r, "s"] \arrow[d, "\sigma"'] & \mathfrak{U} \arrow[dl, "\eta"] \\ \mathfrak{A} & \end{tikzcd}$$

The following theorem gives the main properties of special universal envelopes.

Theorem 1. (1) *If* $(\mathfrak{U}, s)$ *and* $(\mathfrak{U}', s')$ *are special universal envelopes for* $\mathfrak{J}$ *then there exists a unique isomorphism* j *of* $\mathfrak{U}$ *onto* $\mathfrak{U}'$ *such that* $s' = sj$. (2) $\mathfrak{U}$ *is generated by the image* $\mathfrak{J}^s$ *of* $\mathfrak{J}$ *under* s. (3) *There exists a unique involution* π *called the main involution of* $\mathfrak{U}$ *such that* $a^{s\pi} = a^s$, $a \in \mathfrak{J}$. (4) *If* ζ *is homomorphism of* $\mathfrak{J}$ *into a second Jordan algebra* $\mathfrak{J}'$ *and* $(\mathfrak{U}, s)$ $(\mathfrak{U}', s')$ *are special universal envelopes for* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *respectively then there exists a unique homomorphism* ζ_u *of* $\mathfrak{U}$ *into* $\mathfrak{U}'$ *such that the following diagram is commutative:*

(2)
$$\begin{tikzcd} \mathfrak{J} \arrow[r, "\zeta"] \arrow[d, "s"'] & \mathfrak{J}' \arrow[d, "s'"] \\ \mathfrak{U} \arrow[r, "\zeta_u"'] & \mathfrak{U}' \end{tikzcd}$$

(5) *If* $\mathfrak{K}$ *is an ideal in* $\mathfrak{J}$ *and* $\mathfrak{B}$ *is the ideal in* $\mathfrak{U}$ *generated by* $\mathfrak{K}^s$ *then* $\bar{s}$: $a + \mathfrak{K} \to a^s + \mathfrak{B}$ *is an associative specialization of* $\overline{\mathfrak{J}} = \mathfrak{J}/\mathfrak{K}$ *in* $\overline{\mathfrak{U}} = \mathfrak{U}/\mathfrak{B}$ *and* $(\overline{\mathfrak{U}}, \bar{s})$ *is a special universal envelope for* $\overline{\mathfrak{J}}$. (6) *If* P *is an extension field of the base field* Φ *of* $\mathfrak{J}$ *and* s *is the linear extension of* s *to* $\mathfrak{J}_P$ *then* $(\mathfrak{U}_P, s)$ *is a special universal envelope for* $\mathfrak{J}_P$. (7) *If* D *is a derivation in* $\mathfrak{J}$ *then there exists a unique derivation* D_u *in* $\mathfrak{U}$ *such that* $Ds = sD_u$. (8) *If* $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ *where* $\mathfrak{J}_i$ *is an ideal in* $\mathfrak{J}$ *and* $(\mathfrak{U}_i, s_i)$ *is a special universal envelope for* $\mathfrak{J}_i$ *then* $(\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_2,\ s = s_1 + s_2)$ *(that is,* $(a_1 + a_2)^s = a_1^{s_1} + a_2^{s_2}$*) is a special universal envelope for* $\mathfrak{J}$. (9) *Let* $\mathfrak{J}$ *be a Jordan algebra which is a direct limit of the Jordan algebras* $\mathfrak{J}_\alpha$ *defined by the homomorphisms* $\varphi_{\alpha\beta}$ *for* $\alpha < \beta$ *and let* $(\mathfrak{U}_\alpha, s_\alpha)$ *be a special universal envelope for* $\mathfrak{J}_\alpha$. *Let* $\Phi_{\alpha\beta}$ *be the homomorphism of* $\mathfrak{U}_\alpha$ *into* $\mathfrak{U}_\beta$ *such that* $s_\alpha \Phi_{\alpha\beta} = \varphi_{\alpha\beta} s_\beta$. *Then the* $\Phi_{\alpha\beta}$ *define a direct limit* $\mathfrak{U}$ *of the associative algebras* $\mathfrak{U}_\alpha$. *Moreover, if* φ_α *and* Φ_α *denote the canonical homomorphisms of* $\mathfrak{J}_\alpha$ *into* $\mathfrak{J}$ *and of* $\mathfrak{U}_\alpha$ *into* $\mathfrak{U}$ *respectively, then there exists a unique associative specialization* s *of* $\mathfrak{J}$ *in* $\mathfrak{U}$ *such that* $\varphi_\alpha s = s_\alpha \Phi_\alpha$ *and* $(\mathfrak{U}, s)$ *is a special universal envelope for* $\mathfrak{J}$. (10) *If the dimensionality* $\dim \mathfrak{J}/\Phi = n < \infty$ *and* $(\mathfrak{U}, s)$ *is a special universal envelope for* $\mathfrak{J}$ *then* $\dim \mathfrak{U} \leqslant 2^{n-1}$.

The proofs of (1)–(5) and (7) are exactly like the corresponding ones for the Birkhoff–Witt algebra of a Lie algebra given in the author's *Lie Algebras*, pp. 152–157. Hence we omit these. To prove (6) we suppose we have an associative specialization σ of $\mathfrak{J}_P$ in an associative algebra $\mathfrak{A}/P$. Considering $\mathfrak{J}$ as a Φ-subalgebra of $\mathfrak{J}_P$ and $\mathfrak{A}$ as algebra over Φ then the restriction $\sigma|\mathfrak{J}$ of σ to $\mathfrak{J}$ is an associative specialization of $\mathfrak{J}/\Phi$ in $\mathfrak{A}/\Phi$. Hence there exists a homomorphism η of $\mathfrak{U}$ into $\mathfrak{A}/\Phi$ such that $s\eta = \sigma|\mathfrak{J}$. Since $\mathfrak{A}$ is also an algebra over P we can extend η to a homomorphism η of $\mathfrak{U}_P$ into $\mathfrak{A}$. Also the linear extension s of s to $\mathfrak{J}_P$ is an associative specialization of $\mathfrak{J}_P$ in $\mathfrak{U}_P$. Since $s\eta = \sigma|\mathfrak{J}$ holds in $\mathfrak{J}$, $s\eta = \sigma$ holds in $\mathfrak{J}_P$. Hence we have proved the existence of a homomorphism η of $\mathfrak{U}_P$ into $\mathfrak{A}$ such that $s\eta = \sigma$ for the given associative specialization σ of $\mathfrak{J}_P$ in $\mathfrak{A}$. Since $\mathfrak{J}^s$ generates $\mathfrak{U}$, $\mathfrak{J}_P{}^s$ generates $\mathfrak{U}_P$. Hence η is unique. Thus $(\mathfrak{U}_P, s)$ is a special universal envelope for $\mathfrak{J}_P$. The direct sum property stated in (8) is proved in N. Jacobson and F. D. Jacobson's paper [1], p. 147. The property of commutativity with direct limits given in (9) is a standard one for universal mappings (see Milnor and Moore [1], p. 239). Hence we omit the proof of this. Property (10) is proved in F. D. Jacobson and N. Jacobson [1], p. 145 and in Birkhoff and Whitman [1], p. 122. More generally, it is easy to see that if $\{1, u_\alpha\}$, where the indices α take on values in an ordered set I, is a basis for $\mathfrak{J}$ then every element of $\mathfrak{U}$ is a linear combination of 1 and the monomials $u_{\alpha_1} u_{\alpha_2} \dots u_{\alpha_r}$ where $\alpha_1 < \alpha_2 < \dots < \alpha_r$.

Examples. (1) Let $\mathfrak{M}$ be a vector space over Φ, Q a quadratic form on $\mathfrak{M}$ and let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{M}$ a direct sum of $\mathfrak{M}$ and a one-dimensional space with 1 as basis. If $\alpha, \beta \in \Phi$ and $x, y \in \mathfrak{M}$ then we define $(\alpha 1 + x)\cdot(\beta 1 + y) = (\alpha\beta + (x, y))1 + \alpha y + \beta x$ where $(x, y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y))$. Then $\mathfrak{J}$ is a Jordan algebra called the *Jordan algebra of the quadratic form Q*. Let $C(\mathfrak{M}, Q)$ denote the Clifford algebra of Q and identify $\mathfrak{M}$ as usual with a subspace of $C(\mathfrak{M}, Q)$ (Chevalley, *Algebraic Theory of Spinors*, p. 37). Then it is easily seen that $C(\mathfrak{M}, Q)$ and s defined as $(\alpha 1 + x)^s = \alpha 1 + x$ is a special universal envelope for $\mathfrak{J}$. (2) Let $FJ^{(r)}$ denote the free Jordan algebra with r (free) generators $x_1, x_2, \dots, x_r$ and $FA^{(r)}$ the free associative algebra with r (free) generators $u_1, u_2, \dots, u_r$. Let s be the associative specialization of $FJ^{(r)}$ in $FA^{(r)}$ such that $x_i{}^s = u_i$, $i = 1, 2, \dots, r$. Then it follows directly from the definition and the defining property of $FA^{(r)}$ that $(FA^{(r)}, s)$ is special universal envelope for $FJ^{(r)}$. The main involution in $FA^{(r)}$ is the reversal (linear) operator, which maps a monomial $u_{i_1} u_{i_2} \dots u_{i_k}$ into its reverse $u_{i_k} u_{i_{k-1}} \dots u_{i_1}$. The image $FJ^{(r)s}$ in $FA^{(r)}$ is the subalgebra of $FA^{(r)+}$ generated by the u_i. This is called the free special Jordan algebra with r generators u_i.

The two examples just given indicate that special universal envelopes may be obtained by different constructions. We shall now give one which we shall take as our standard construction. Let $\mathfrak{J}$ be a given Jordan algebra and let $\mathfrak{T}(\mathfrak{J})$ be the tensor algebra based on the vector space $\mathfrak{J}$: $\mathfrak{T}(\mathfrak{J}) = \Phi 1 \oplus \mathfrak{J} \oplus (\mathfrak{J} \otimes \mathfrak{J}) \oplus \dots$. Let $\mathfrak{R}$ be the ideal in $\mathfrak{T}(\mathfrak{J})$ generated by the elements of the form $a \cdot b - \frac{1}{2}(a \otimes b + b \otimes a)$, $1_{\mathfrak{J}} - 1$ where $a, b \in \mathfrak{J}$ and $1_{\mathfrak{J}}$ is the identity element of $\mathfrak{J}$ and set $\mathrm{su}(\mathfrak{J}) = \mathfrak{T}(\mathfrak{J})/\mathfrak{R}$. Then it is straight-forward to check that $(\mathrm{su}(\mathfrak{J}), s)$ where $a^s = a + \mathfrak{R}$, $a \in \mathfrak{J}$, is a special universal envelope for $\mathfrak{J}$. We shall refer to this one as *the* special universal envelope for $\mathfrak{J}$. As before, we let π denote the main involution in $\mathrm{su}(\mathfrak{J})$.

We now have a mapping $\mathfrak{J} \to (\mathrm{su}(\mathfrak{J}), \pi)$ of Jordan algebras over a field Φ into associative algebras with involution over Φ. Let ζ be a homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$. Then we have a unique homomorphism ζ_u of su $(\mathfrak{J})$ into su $(\mathfrak{J}')$ such that $\zeta s' = s\zeta_u$ where s and s' are the given associative specializations of $\mathfrak{J}$ and $\mathfrak{J}'$ in su $(\mathfrak{J})$ and su $(\mathfrak{J}')$ respectively. If π' denotes the main involution su $(\mathfrak{J}')$ then we have $a^{s\pi\zeta_u} = a^{s\zeta_u} = a^{\zeta s'}$ and $a^{s\zeta_u\pi'} = a^{\zeta s'}$, for $a \in \mathfrak{J}$. Since $\mathfrak{J}^s$ generates su $(\mathfrak{J})$ we have $\zeta_u\pi' = \pi\zeta_u$ so ζ_u is a homomorphism of (su $(\mathfrak{J}), \pi$) into (su $(\mathfrak{J}'), \pi'$). Next let λ be a homomorphism of $\mathfrak{J}'$ into $\mathfrak{J}''$ then it is clear from Theorem 1 (4) that $(\zeta\lambda)_u = \zeta_u\lambda_u$. It follows that the mappings $\mathfrak{J} \to$ (su $(\mathfrak{J}), \pi$), $\zeta \to \zeta_u$ define a functor S from cat J/Φ to cat AI/Φ.

2. *General theorems.* Let $\mathfrak{J}/\Phi$ be a Jordan algebra and let s be the associative specialization of $\mathfrak{J}$ into the special universal envelope su $(\mathfrak{J})$. Since $a^{s\pi} = a^s$, $a \in \mathfrak{J}$, $\mathfrak{J}^s \subseteq \mathfrak{H}(\text{su}\,(\mathfrak{J}), \pi)$. Clearly s is a homomorphism of $\mathfrak{J}$ into the Jordan algebra $\mathfrak{H}(\text{su}\,(\mathfrak{J}), \pi)$. It is clear also from the definition that $\mathfrak{J}$ is a special Jordan algebra, that is, $\mathfrak{J}$ has an injective associative specialization if and only if s is injective. We shall now call $\mathfrak{J}$ a *reflexive* Jordan algebra if s is surjective on $\mathfrak{H}(\text{su}\,(\mathfrak{J}), \pi)$. Then $\mathfrak{J}$ is special and reflexive if and only if s is an isomorphism of $\mathfrak{J}$ onto $\mathfrak{H}(\text{su}\,(\mathfrak{J}), \pi)$.

Next let $(\mathfrak{A}, J)$ be an associative algebra with involution over the field Φ and let $\mathfrak{H}(\mathfrak{A}, J)$ be the Jordan algebra of J-symmetric elements. Then the injection mapping $a \to a$ of $\mathfrak{H}(\mathfrak{A}, J)$ is an associative specialization of $\mathfrak{H}(\mathfrak{A}, J)$ in $\mathfrak{A}$. Hence we have a unique homomorphism h of su $(\mathfrak{H}(\mathfrak{A}, J))$ into $\mathfrak{A}$ such that $a^{sh} = a$, $a \in \mathfrak{H}(\mathfrak{A}, J)$. We have $a^{s\pi h} = a$ and $a^{shJ} = a$. Since the a^s, $a \in \mathfrak{H}(\mathfrak{A}, J)$, generate su (J) it is clear that h is a homomorphism of (su $(J), \pi$) into $(\mathfrak{A}, J)$. It is clear also that h is surjective if and only if $\mathfrak{H}(\mathfrak{A}, J)$ generates $\mathfrak{A}$. We shall now call $(\mathfrak{A}, J)$ a *perfect* associative algebra with involution if h is an isomorphism of (su $(\mathfrak{J}), \pi$) onto $(\mathfrak{A}, J)$. This is the case if and only if $\mathfrak{A}$ together with the injection mapping of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{A}$ is a special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)$. More explicitly, $(\mathfrak{A}, J)$ is perfect if and only if any associative specialization of $\mathfrak{H}(\mathfrak{A}, J)$ has a unique extension to a homomorphism of $\mathfrak{A}$. It is clear from the definitions that if $(\mathfrak{A}, J)$ is perfect then $\mathfrak{H}(\mathfrak{A}, J)$ is special and reflexive.

We shall now give some sufficient conditions for perfection and reflexivity.

Theorem 2. *Any Jordan algebra with* $\leqslant 3$ *generators is reflexive.*

Proof. We have seen that if $FJ^{(r)}$ is the free Jordan algebra with r generators $x_1, x_2, \ldots, x_r$ and $FA^{(r)}$ is the free associative algebra with r generators $u_1, u_2, \ldots, u_r$ then $FA^{(r)}$ together with the associative specialization such that $x_i \to u_i$ constitute a special universal envelope for $FJ^{(r)}$. We have seen also that the main involution in $FA^{(r)}$ is the reversal operator. It has been proved by P. Cohn that the space of symmetric elements under the reversal operator coincides with the Jordan subalgebra of $FA^{(r)+}$ generated by the u_i if and only if $r \leqslant 3$ (Cohn [1], pp. 257–259).

It follows that $FJ^{(r)}$ is reflexive if and only if $r \leqslant 3$. The proof of Theorem 2 will now follow by showing that if $\mathfrak{J}$ is reflexive then any homomorphic image of $\mathfrak{J}$ is reflexive. Let ζ be a homomorphism of $\mathfrak{J}$ onto $\overline{\mathfrak{J}}$. Since the images of $\mathfrak{J}$ and $\overline{\mathfrak{J}}$ in su $(\mathfrak{J})$ and su $(\overline{\mathfrak{J}})$ generate these algebras it is clear that the corresponding homomorphism ζ_u of su $(\mathfrak{J})$ is surjective on su $(\overline{\mathfrak{J}})$. Let $\bar{a} \in \mathfrak{H}(\mathrm{su}\,(\overline{\mathfrak{J}}), \bar{\pi})$, $\bar{\pi}$ the main involution in su $(\overline{\mathfrak{J}})$. Then $\bar{a} = \frac{1}{2}(\bar{a} + \bar{a}^{\bar{\pi}})$ and $\bar{a} = a^{\zeta_u}$ for some $a \in \mathrm{su}\,(\mathfrak{J})$. Hence $\bar{a} = \frac{1}{2}(\bar{a} + \bar{a}^{\bar{\pi}}) = \frac{1}{2}(a^{\zeta_u} + a^{\zeta_u \bar{\pi}}) = \frac{1}{2}(a^{\zeta_u} + a^{\pi \zeta_u}) = b^{\zeta_u}$ where $b = \frac{1}{2}(a + a^{\pi}) \in \mathfrak{H}(\mathrm{su}\,(\mathfrak{J}), \pi)$. If $\mathfrak{J}$ is reflexive then $b \in \mathfrak{J}^s$. Then $b = c^s$, $c \in \mathfrak{J}$ and $\bar{a} = c^{s\zeta_u} = c^{\zeta\bar{s}} \in \overline{\mathfrak{J}}^{\bar{s}}$, $\bar{s}$ the associative specialization of $\overline{\mathfrak{J}}$ in su $(\overline{\mathfrak{J}})$. Hence $\overline{\mathfrak{J}}$ is reflexive.

It is a known result due to Shirshov and Cohn (Shirshov [1], p. 159) that every Jordan algebra with $\leqslant 2$ generators is special. Together with Theorem 2 we have that any such algebra is special and reflexive. Thus such an algebra $\mathfrak{J}$ is isomorphic to the Jordan algebra $\mathfrak{H}(\mathrm{su}\,(\mathfrak{J}), \pi)$. Consequently, any Jordan algebra with two generators is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ for some associative algebra with involution $(\mathfrak{A}, J)$. Clearly this is a strengthening of the Theorem of Shirshov–Cohn.

Theorem 3. *For any Jordan algebra $\mathfrak{J}$ the associative algebra with involution* (su $(\mathfrak{J}), \pi$) *is perfect.*

Proof. Set $\mathfrak{U} = \mathrm{su}\,(\mathfrak{J})$, $\mathfrak{K} = \mathfrak{H}(\mathfrak{U}, \pi)$, $\mathfrak{B} = \mathrm{su}\,(\mathfrak{K})$ and let s, t denote the given associative specializations of $\mathfrak{J}$ and $\mathfrak{K}$ in $\mathfrak{U}$ and $\mathfrak{B}$ respectively. Then $\mathfrak{J}^s$ generates $\mathfrak{U}$ and $\mathfrak{K}^t$ generates $\mathfrak{B}$. Hence $\mathfrak{J}^{st}$ generates $\mathfrak{B}$. Considering s as a homomorphism of the Jordan algebra $\mathfrak{J}$ into the Jordan algebra $\mathfrak{K}$ we obtain a homomorphism s_u of $\mathfrak{U}$ into $\mathfrak{B}$ such that

$$\begin{tikzcd} \mathfrak{J} \arrow[r, "s"] \arrow[d, "s"'] & \mathfrak{K} \arrow[d, "t"] \\ \mathfrak{U} \arrow[r, "s_u"'] & \mathfrak{B} \end{tikzcd}$$

is commutative. Then if $a \in \mathfrak{J}$ we have $a^{st} = a^{ss_u}$. Let k be the homomorphism of $\mathfrak{B}$ into $\mathfrak{U}$ such that $b^{tk} = b$, $b \in \mathfrak{K} = \mathfrak{H}(\mathfrak{U}, \pi)$. We have to show that k is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{U}$. Now if $a \in \mathfrak{J}$ then $a^s \in \mathfrak{J}^s \subseteq \mathfrak{H}(\mathfrak{U}, \pi) = \mathfrak{K}$. Hence $a^s = a^{stk} = a^{ss_u k}$. Since the a^s, $a \in \mathfrak{J}$, generate $\mathfrak{U}$ this implies that $s_u k = 1_{\mathfrak{U}}$ the identity mapping on $\mathfrak{U}$. Next, if $a \in \mathfrak{J}$, then $a^{stks_u} = a^{ss_u} = a^{st}$. Since the elements a^{st}, $a \in \mathfrak{J}$, generate $\mathfrak{B}$ this implies that $ks_u = 1_{\mathfrak{B}}$. Hence k is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{U}$.

In the next section we shall give an improved formulation and new derivation of the main theorem on the classification of special finite dimensional central simple Jordan algebras (cf. F. D. Jacobson and N. Jacobson [1], pp. 157–167). For this a special case of the following theorem due to Jacobson and Rickart ([1], p. 315) will be fundamental.

Theorem 4. *Let $\mathfrak{D}$ be an associative algebra with an involution $d \to \bar{d}$, $\mathfrak{D}_n$, $n \geqslant 3$, the $n \times n$ matrix algebra over $\mathfrak{D}$, J_a the involution $X \to a^{-1}\bar{X}'a$ where $\bar{X}'$ is the transpose of $\bar{X} = (\bar{x}_{ij})$, $X = (x_{ij})$, and a is a diagonal matrix* diag $\{a_1, a_2, \dots, a_n\}$ *where $\bar{a}_i = a_i$ has an inverse in $\mathfrak{D}$. Then $(\mathfrak{D}_n, J_a)$ is perfect.*

We remark that the result of Jacobson–Rickart is more general than the foregoing in that $\mathfrak{D}$ can be an arbitrary associative ring rather than an algebra. We note also that this result has been generalized recently by W. Martindale ([1]) and Martindale's theorem has the following immediate consequence.

Theorem 5. *Let $(\mathfrak{A}, J)$ be an associative algebra with an involution having the following properties: $\mathfrak{A}$ contains three J-symmetric idempotent elements such that $e_i e_j = 0$, $i \neq j$, $e_1 + e_2 + e_3 = 1$ and $\mathfrak{A} e_i \mathfrak{A} = \mathfrak{A}$, $i = 1, 2, 3$. Then $(\mathfrak{A}, J)$ is perfect.*[2])

To obtain Theorem 4 from Theorem 5 we remark first that the latter result has an immediate extension in which the hypothesis on the three idempotents is replaced by one on $n \geqslant 3$ idempotents. For, suppose $e_1, e_2, \dots, e_n$ satisfy the conditions $e_i^2 = e_i = e_i^J$, $e_i e_j = 0$, $i \neq j$, $\sum e_i = 1$, $\mathfrak{A} e_i \mathfrak{A} = \mathfrak{A}$. Set $f_1 = e_1$, $f_2 = e_2$, $f_3 = \sum\limits_{i > 2} e_i$. Then the f_i satisfy the conditions of the theorem. Hence $(\mathfrak{A}, J)$ is perfect. In particular, let $\mathfrak{A} = \mathfrak{D}_n$, $J = J_a$ as in Theorem 4. Then the diagonal idempotents $e_i =$ diag $\{0, \dots, 0, 1, 0, \dots, 0\}$ satisfy the indicated conditions. Hence $(\mathfrak{D}_n, J_a)$ is perfect.

3. *Classification of finite dimensional special central simple Jordan algebras.* Let $(u_1, u_2, \dots, u_n)$ be a basis for a Jordan algebra $\mathfrak{J}/\Phi$ and let $\Xi = \Phi(\xi_1, \xi_2, \dots, \xi_n)$ be the field of rational expressions in n indeterminates ξ_i. The element $x = \sum\limits_1^n \xi_i u_i$ of $\mathfrak{J}_P$ is called a generic element of $\mathfrak{J}$ and the degree of the minimum polynomial $m_x(\lambda)$ of x is called the degree of $\mathfrak{J}$ (Jacobson [4], p. 179). It is easily seen that the degree is independent of the basis and $\mathfrak{J}$ and $\mathfrak{J}_P$ have the same degree for every extension field P/Φ. Moreover, if $\mathfrak{J}$ is simple over an algebraically closed field then the degree of $\mathfrak{J}$ is the maximum cardinality for sets of non-zero orthogonal idempotent elements of $\mathfrak{J}$.

The finite dimensional simple Jordan algebras over an algebraically closed field Ω have been determined by Albert ([1], p. 252, [2], p. 567, [3], p. 524, also Jacobson [2]). His results can be formulated as follows. If $\mathfrak{J}/\Omega$ is simple and the degree

deg $\mathfrak{J} = 1$, then $\mathfrak{J} = \Omega 1$.

deg $\mathfrak{J} = 2$, then $\mathfrak{J}$ is isomorphic to the Jordan algebra of a non-degenerate quadratic form Q in a vector space $\mathfrak{M}$ with deg $\mathfrak{M} \geqslant 2$,

deg $\mathfrak{J} = n \geqslant 3$, then $\mathfrak{J}$ is isomorphic to a Jordan algebra $\mathfrak{H}(\mathfrak{D}_n) \equiv$

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[2]) Martindale's hypotheses are different but equivalent to the foregoing in that the condition $\mathfrak{A} e_i \mathfrak{A} = \mathfrak{A}$ is replaced by $e_i \mathfrak{A} e_j \mathfrak{A} e_i = e_i \mathfrak{A} e_i$, $i \neq j$.

$\equiv \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $\mathfrak{D}$ is a composition algebra and the involution $\mathfrak{J}_1$ is $X \to \bar{X}'$, $d \to \bar{d}$ the standard involution in $\mathfrak{D}$. Moreover, $\mathfrak{D}$ is associative if $n > 3$. We recall that a composition algebra is an algebra equipped with a non-degenerate quadratic form Q such that $Q(xy) = Q(x)Q(y)$. Such an algebra is necessarily alternative and has a uniquely determined involution $d \to \bar{d}$, called the standard involution, such that $d\bar{d} = Q(d)1 = \bar{d}d$ (Jacobson [3]). In the algebraically closed case the possibilities for $\mathfrak{D}$ and its standard involution are the following: I. $\mathfrak{D} = \Omega 1$, involution the identity mapping, II. $\mathfrak{D} = \Omega 1_1 \oplus \Omega 1_2$, involution exchanging the two ideals $\Omega 1_i$, III. $\mathfrak{D} = \Omega_2$, involution $d \to tr\,(d)1 - d$ where $tr\, d$ is the trace of the matrix d, IV. $\mathfrak{D}$ a split Cayley algebra with its standard involution.

Since IV is not associative it can be used only if $n = 3$. The resulting algebra is an exceptional (non-special) Jordan algebra. All the other Jordan algebras in the list are special. Also any Jordan algebra of the type listed above is simple and has the indicated degree.

An associative algebra with involution $(\mathfrak{A}, J)$ is *simple* if it has no ideals other than $\mathfrak{A}$ and 0 and $\mathfrak{A} \neq 0$. If $(\mathfrak{A}, J)$ is simple then either $\mathfrak{A}$ is simple or $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is a simple ideal. It is well known that any finite dimensional simple associative algebra with involution over an algebraically closed field Ω is isomorphic to one of the following algebras with involution: *A.* $\Omega_n \oplus \Omega_n$, involution $(X, Y) \to (Y', X')$, $X, Y \in \Omega_n$, X', Y' the transposes of X, Y. *B.* Ω_n, involution $X \to X'$, *C.* Ω_{2n}, involution $X \to S^{-1}X'S$ where $S = \mathrm{diag}\,\{q, \ldots, q\}$, $q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is easily seen that the algebras listed can be described in a more uniform way as the algebras $\mathfrak{D}_n$, $\mathfrak{D}$ an associative composition algebra with the *standard involution* J_1: $X \to \bar{X}'$ in $\mathfrak{D}_n$. The case A corresponds to II in the list of composition algebras over Ω and B and C correspond to I and III respectively. Conversely, any $(\mathfrak{D}_n, J_1)$, $\mathfrak{D}$ a composition algebra over Ω, J_1 the standard involution, is simple. Moreover, two such algebras are isomorphic if and only if the n's indicated are the same and the composition algebras are isomorphic. We shall call the invariant n the *degree* of $(\mathfrak{D}_n, J_1)$.

Albert's result for special algebras of degree $\geqslant 3$ can now be stated in the following way: Any finite dimensional special simple Jordan algebra of degree $n \geqslant 3$ over an algebraically closed field Ω is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite dimensional simple associative algebra of degree n over Ω and conversely. We shall now consider the extension of this result to arbitrary base fields.

We recall that a Jordan algebra $\mathfrak{J}/\Phi$ is central simple if $\mathfrak{J}$ is simple and the only elements c in $\mathfrak{J}$ such that $[x, y, c] \equiv (x \cdot y) \cdot c - x \cdot (y \cdot c) = 0 = [x, c, y] = [x, y, c]$ are the elements of $\Phi 1$. A simple algebra over an algebraically closed field is necessarily central simple. $\mathfrak{J}/\Phi$ is central simple if and only if $\mathfrak{J}_P$ is simple for every extension field P/Φ. If $\mathfrak{J}$ is central simple of degree 1 then $\mathfrak{J}\Omega = \Omega 1$ if Ω is the algebraic closure of the base field Φ. Hence $\mathfrak{J}=\Phi 1$. If $\mathfrak{J}$ is central simple of degree 2 then $\mathfrak{J}_\Omega$ is isomorphic to the Jordan algebra of a non-degenerate quadratic form on a vector space $\mathfrak{M}$ with $\dim \mathfrak{M}>1$. It follows easily that $\mathfrak{J}$ is of the same type and the classification of these algebras is equivalent to the classification of non-degenerate quadratic forms. The central simple Jordan algebras of degree $\geqslant 3$ include exceptional ones. Our considerations play no role in the study of these algebras. (For these the main references are ALBERT–JACOBSON [1], ALBERT [4], SPRINGER [1].) We shall now consider the problem of classifying the finite dimensional special central simple Jordan algebras of degree $\geqslant 3$.

We define the *center* $\mathfrak{C}_J$ of an associative algebra with involution $(\mathfrak{A}, J)$ to be $\mathfrak{C}_J=\mathfrak{C} \cap \mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{C}$ is the center of $\mathfrak{A}$. An associative algebra with involution is *central simple* if it is simple and its center is $\Phi 1$ (Φ the base field). It is easy to see that $(\mathfrak{A}, J)$ is central simple if and only if $(\mathfrak{A}_P, J)$ is simple for every extension field P/Φ. Also if $\mathfrak{A}$ is finite dimensional over an algebraically closed field Ω and $(\mathfrak{A}, J)$ is simple then $(\mathfrak{A}, J)$ is central simple and we have noted that such an algebra with involution is isomorphic to one of the pairs $(\mathfrak{D}_n, J_1)$ where $\mathfrak{D}$ is a composition algebra and J_1 is the standard involution. If $(\mathfrak{A}, J)$ is finite dimensional central simple over Φ and Ω is the algebraic closure of Φ then $(\mathfrak{A}_\Omega, J)$ is isomorphic to one of the algebras $(\mathfrak{D}_n, J_1)$. We shall call the invariant n the *degree* of $(\mathfrak{A}, J)$.

We shall call a property $\mathfrak{A}$ of Jordan algebras (associative algebras with involution) *linear* if the validity of A for $\mathfrak{J}((\mathfrak{A}, J))$ is equivalent to its validity for $\mathfrak{J}_P((\mathfrak{A}_P, J))$, P any extension field of Φ. Clearly the property that the degree is a fixed integer n is linear. Also it is easily seen that the property of central simplicity is linear. We also have the following

Lemma. *The property that Jordan algebras are special or reflexive is linear. The property that associative algebras with involution are perfect is linear.*

Proof. Suppose $\mathfrak{J}$ is special (reflexive). Then the associative specialization s of $\mathfrak{J}$ in su $(\mathfrak{J})$ is injective (surjective). Hence the linear extension s of s to $\mathfrak{J}_P$ into su $(\mathfrak{J})_P$ is injective (surjective). Since (su $(\mathfrak{J})_P, s)$ is a special universal envelope for $\mathfrak{J}_P$ it follows that the given associative specialization s of $\mathfrak{J}_P$ in su $(\mathfrak{J}_P)$ is injective (surjective). Hence $\mathfrak{J}_P$ is special (reflexive). Conversely, assume $\mathfrak{J}_P$ is special (reflexive). Then a reversal of the steps of the argument shows that $\mathfrak{J}$ is special (reflexive). Next assume $(\mathfrak{A}, J)$ is perfect. Then $\mathfrak{A}$ and the injection mapping of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{A}$ constitute a special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)$. Hence by Theorem 1 (6), $\mathfrak{A}_P$ and the canonical isomorphism of $\mathfrak{H}(\mathfrak{A}, J)_P$ in $\mathfrak{A}_P$ constitute a special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)_P$. Hence $\mathfrak{A}_P$ and the injection mapping form a special universal envelope for $\mathfrak{H}(\mathfrak{A}_P, J)$ and $(\mathfrak{A}_P, J)$ is perfect. Conversely, assume $(\mathfrak{A}_P, J)$ perfect and let σ be an associative specialization of $\mathfrak{H}(\mathfrak{A}, J)$ in an associative algebra $\mathfrak{B}$. Then the linear extension of σ to $\mathfrak{H}(\mathfrak{A}, J)_P = \mathfrak{H}(\mathfrak{A}_P, J)$ can be extended to a homomorphism of $\mathfrak{A}_P$ into $\mathfrak{B}_P$. This implies that σ can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Since $\mathfrak{H}(\mathfrak{A}_P, J)$ generates $\mathfrak{A}_P$, $\mathfrak{H}(\mathfrak{A}, J)$ generates $\mathfrak{A}$. Hence the extension of σ to $\mathfrak{A}$ is unique. Hence $\mathfrak{A}$ and the injection mapping form a special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)$ so $(\mathfrak{A}, J)$ is perfect.

We can now prove the following key result for the classification of special central simple Jordan algebras.

Theorem 6. (1) *Let $(\mathfrak{A}, J)$ be a finite dimensional central simple associative algebra of degree $n \geqslant 3$. Then $(\mathfrak{A}, J)$ is perfect and $\mathfrak{H}(\mathfrak{A}, J)$ is a finite dimensional central simple Jordan algebra of degree n.* (2) *Let $\mathfrak{J}$ be a finite dimensional special central simple Jordan algebra of degree $n \geqslant 3$. Then $\mathfrak{J}$ is reflexive and $(\mathrm{su}\,(\mathfrak{J}), \pi)$ is a finite dimensional central simple associative algebra with involution of degree n.*

Proof. The results we have noted show that all the properties stated in the hypotheses and conclusions of the theorem are linear. Hence it suffices to prove the theorem for an algebraically closed base field Ω. Let $(\mathfrak{A}, J)$ be a finite dimensional central simple associative algebra with involution over Ω of degree $n \geqslant 3$. Then $(\mathfrak{A}, J)$ is isomorphic to $(\mathfrak{D}_n, J_1)$, $\mathfrak{D}$ an associative composition algebra. Then $(\mathfrak{D}_n, J_1)$, hence $(\mathfrak{A}, J)$ is perfect by Theorem 4. Also $\mathfrak{H}(\mathfrak{D}_n) \equiv \mathfrak{H}(\mathfrak{D}_n, J_1)$ is central simple of degree n, so the same is true of $\mathfrak{H}(\mathfrak{A}, J)$. Next let $\mathfrak{J}$ be a finite dimensional special central simple Jordan algebra of degree $n \geqslant 3$ over Ω. Then $\mathfrak{J}$ is isomorphic to a Jordan algebra $\mathfrak{H}(\mathfrak{D}_n)$, $\mathfrak{D}$ an associative composition algebra, by Albert's theorem. Since $(\mathfrak{D}_n, J_1)$ is perfect, $\mathfrak{H}(\mathfrak{D}_n)$ is reflexive and $\mathfrak{J}$ is reflexive. Also $(\mathrm{su}\,(\mathfrak{J}), \pi)$ is isomorphic to $(\mathfrak{D}_n, J_1)$ by Theorem 4 and $(\mathfrak{D}_n, J_1)$ is a finite dimensional central simple associative algebra with involution of degree n. Hence the same is true of $(\mathrm{su}\,(\mathfrak{J}), \pi)$.

We remark that the reflexivity of $\mathfrak{J}$ can also be deduced from Theorem 2 since one can show that $\mathfrak{H}(\mathfrak{D}_n)$, $\mathfrak{D}$ a composition algebra, is generated by three elements. One can use this to prove also that any finite dimensional central simple Jordan algebra of degree $\geqslant 3$ can be generated by three elements. The following is the main result on the classification of special central simple Jordan algebras.

Theorem 7. *A Jordan algebra is finite dimensional special central simple of degree $\geqslant 3$ if and only if it is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite dimensional central simple associative algebra with involution of degree $\geqslant 3$. If $(\mathfrak{A}, J)$, $(\mathfrak{B}, K)$ are finite dimensional central simple associative algebras with involutions of degree $\geqslant 3$ then these are isomorphic if and only if the Jordan algebras $\mathfrak{H}(\mathfrak{A}, J)$, $\mathfrak{H}(\mathfrak{B}, K)$ are isomorphic.*

Proof. If $\mathfrak{J}$ is a special finite dimensional central simple Jordan algebra of degree $\geqslant 3$ then $\mathfrak{J}$ is isomorphic to a Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$

where $(\mathfrak{A}, J)$ is a finite dimensional central simple associative algebra with involution of degree $\geqslant 3$, by Theorem 6 (2). The converse follows the second statement in Theorem 6 (1). If $(\mathfrak{B}, K)$ is a second associative algebra with involution of the kind indicated then an isomorphism of $\mathfrak{H}(\mathfrak{A}, J)$ onto $\mathfrak{H}(\mathfrak{B}, K)$ has a unique extension to an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ since $\mathfrak{A}$ and the injection mapping and $\mathfrak{B}$ and the injection mapping are special universal envelopes for $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{H}(\mathfrak{B}, K)$ by Theorem 6 (1). Thus $\mathfrak{H}(\mathfrak{A}, J) \cong \mathfrak{H}(\mathfrak{B}, K)$ implies $(\mathfrak{A}, J) \cong (\mathfrak{B}, K)$. The converse is clear.

This result reduces the problem of classification for the Jordan algebras to the corresponding one on associative algebras with involutions. What the latter amounts to explicitly can be seen in Jacobson [1], pp. 541–551.

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